THE INFLUENCE OF THE HALL EFFECT ON THE CURRENT STRUCTURE IN A PLASMA FLOW WITH A SPATIALLY PERIODIC MAGNETIC FIELD

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The structure of the electric fields and current is studied for stationary plasma flow in an axially symmetric, spatially periodic magnetic field. The problem is solved in the magnetohydrodynamic approximation with allowance for the Hall term in the generalized Ohm's law equation. It is assumed that the magnetic Reynolds number and the interaction parameter are small.

A number of problems in the dynamics of plasmoids passing through an axially symmetric magnetic field have been satisfactorily explained by means of a generalized flexible current loop model [1, 2]. However, this model does not include the Hall effect and the influence of this can be considerable [3, 4]. The solution of the full system of equations in the magnetohydrodynamic approximation is a complex problem. Nevertheless, it is possible to construct a solution for the electric fields and current alone if their effect on the magnetic field and the change in the flow parameters are neglected. Such an approach is possible for sufficiently small values of the magnetic Reynolds number (which is a measure of the deformation of the magnetic field by the flow) and the interaction parameter (Stuart number) [5].

From the zeroth approximation in these parameters we can find the solution for the current and fields by studying only one equation of the system – the generalized Ohm's law. This method has earlier been used for investigating the influence of the Hall effect for various particular cases of plasma interaction with axially symmetric magnetic fields [6-9]. In [9], for example, the motion of a plasma was studied for a sign-varying, spatially periodic magnetic field. In this paper the study is generalized to the case of a uniform magnetic field which is modulated by a varying, spatially periodic field and an analysis is made of the structure of the vector lines of current density in the plasma as a function of the modulation depth.

If the gaskinetic pressure is neglected, together with the drift of the ions relative to the neutrals, the generalized Ohm's law takes the form

$$\mathbf{j} = \sigma(\mathbf{E} + [\mathbf{v}, \mathbf{B}]) - k[\mathbf{j}, \mathbf{B}] \tag{1}$$

where

$$\sigma = \frac{\sigma^*}{\sigma_0^*}, \quad \mathbf{j} = \frac{\mathbf{j}^*}{\sigma_0^* v_0^* B_0^*}, \quad \mathbf{E} = \frac{\mathbf{E}^*}{v_0^* B_0^*}, \quad \mathbf{v} = \frac{\mathbf{v}^*}{v_0^*}, \quad \mathbf{B} = \frac{\mathbf{B}^*}{B_0^*}$$

Here $k = \omega_e \tau_{ei}$ is the Hall parameter, $\mathbf{j}$ is the current density vector, σ is the conductivity, $\mathbf{E}$ is the electric field, $\mathbf{v}$ is the plasma velocity, $\mathbf{B}$ is the magnetic induction vector, ω_e is the electron gyrofrequency, τ_{ei} is the time between electron-ion collisions; the star denotes dimensional quantities and the subscript 0 denotes quantities for an undisturbed plasma. The magnetic field is characterized by the parameter B_0^* which is the amplitude value of the modulating field on the axis of symmetry. Since we are

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taking all parameters describing the stream (apart from the electric fields and currents) as being undisturbed, the flow velocity v is a constant:

$$\mathbf{v} = \mathbf{e}_z \quad (r \leqslant b)$$

where $\mathbf{e}_z$ is the unit vector along the axis of symmetry (the z axis) and b is the radious of the infinite plasma cylinder; the conductivity σ is equal to unity inside the cylinder and zero outside.

We consider the zeroth approximation in the small magnetic Reynolds number and, therefore, the external magnetic field can be considered as undeformed. It is assumed that the spatially periodic magnetic field $\mathbf{B}(B_r, O, B_z)$ is given by

$$B_r = I_1(r)\cos z, \quad B_z = -I_0(r)\sin z - B_1 \tag{2}$$

where $I_0(r)$ and $I_1(r)$ are modified Bessel functions, $r = r^*/L^*$, $z = z^*/L^*$, $L^* = \lambda/2\pi$, and λ is the period of the magnetic field.

Thus to this approximation the unknown quantities in (1) are the electric field E and the current density j. We try for the solution in the form of a power series in the Hall parameter k ($k < 1$):

$$\mathbf{j} = \sum_{n=0}^{\infty} k^n \mathbf{j}_n \tag{3}$$

and similarly for the field E; the expansion coefficients $\mathbf{j}_n$ and $\mathbf{E}_n$ will be called the currents and fields of order n. Successive consideration of the chain of algebraic equations for $\mathbf{j}_n$ and $\mathbf{E}_n$ shows [6] that principally a zero-order current $\mathbf{j}_0$ is induced by the external magnetic field in the absence of a Hall effect and has only an azimuthal component whereas the current $\mathbf{j}_1$ has an azimuthal component. In other words, in the first approximation the Hall current does not represent a correction to the main current but is separate from it on account of the symmetry of the problem. Further approximations give higher-order corrections in k to the main current $\mathbf{j}_0$ ($k^2\mathbf{j}_2$ etc.) and to the Hall current $k\mathbf{j}_1$ ($k^3\mathbf{j}_3$ etc.) We limit ourselves to the first-order field and current.

The equation relating $\mathbf{j}_1$ and $\mathbf{E}_1$ has the form

$$\mathbf{j}_1 = \sigma\mathbf{E}_1 = -\mathbf{F}_0 \tag{4}$$

where

$$\mathbf{F}_0 = [\mathbf{j}_0, \mathbf{B}] = -\sigma[[\mathbf{v}, \mathbf{B}], \mathbf{B}] \tag{5}$$

We transform (4) to an equation for the electric field potential Φ_1 defined by

$$\mathbf{E}_1 = -\nabla\Phi_1 \tag{6}$$

To do this, we find the divergence of both sides of (4). Since $\sigma = 1$ inside the plasma, we get

$$\Delta\Phi_1 = -\operatorname{div}\mathbf{F}_0 \tag{7}$$

The normal component of the current is equal to zero on the free surface of the plasma and thus the potential Φ_1 must satisfy the boundary condition

$$\left.\frac{\partial\Phi_1}{\partial r}\right|_{r=b} = -F_{0r}\big|_{r=b} \tag{8}$$

Since the plasma moves in a spatially periodic magnetic field, the currents can be expected to be periodic functions of z. Hence the derivatives of the potential will satisfy the periodic conditions

$$\frac{\partial\Phi_1(r, z)}{\partial r} = \frac{\partial\Phi_1(r, z+2\pi)}{\partial r}, \quad \frac{\partial\Phi_1(r, z)}{\partial z} = \frac{\partial\Phi_1(r, z+2\pi)}{\partial z}. \tag{9}$$

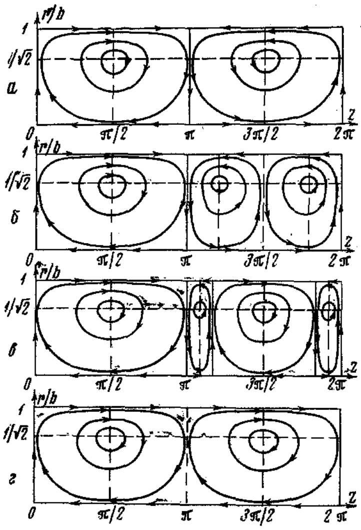

Fig. 1

In order to solve (7) we introduce a new unknown Φ such that instead of the boundary condition (8) we have the following homogeneous condition:

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=b} = 0 \tag{10}$$

where

$$\Phi = \Phi_1 + \int_0^r F_{0r}\, dr \tag{11}$$

With this substitution, Eq. (7) takes the form

$$\Delta\Phi = -\frac{\partial F_{0z}}{\partial z} + \frac{\partial^2}{\partial z^2}\int_0^r F_{0r}\, dr \tag{12}$$

The solution of this equation satisfying (10) can conveniently be found as a Dini expansion in zero-order Bessel functions. After finding Φ (r, z) and transferring to the potential Φ_1(r, z), we finally arrive with the help of (11) at the result

$$\Phi_1(r, z) = 2Cb^{-2}(z - z_0) + \Lambda_1(r)\sin 2z + \Lambda_2(r)\cos z \tag{13}$$

where C and z_0 are constants,

$$\Lambda_1(r) = \frac{2}{b^2}\left[\frac{1}{4}p_1^*(k_0) + \sum_{i=1}^{\infty}\frac{p_1^*(k_i)}{4 + (k_i/b)^2}\frac{J_0(k_i r/b)}{J_0^2(k_i)}\right] + \frac{1}{2}[I_0^2(r) - 1]$$

$$\Lambda_2(r) = \frac{2}{b^2}\left[p_2^*(k_0) + \sum_{i=1}^{\infty}\frac{p_2^*(k_i)}{1 + (k_i/b)^2}\frac{J_0(k_i r/b)}{J_0^2(k_i)}\right] + B_1[I_0(r) - 1]$$

$p_{1,2}^*$ (k_i) are the Dini coefficients of the functions $p_{1,2}(r)$:

$$p_1(r) = I_1^2(r) - I_0^2(r) + 1,\quad p_2(r) = -B_1[I_0(r) - 1]$$

Since the potential is in general defined to within an additive constant, the value of z_0 remains undetermined. We thus obtain a family of solutions depending on the constant C for the potential Φ_1(r, z). An additional condition must be imposed on the potential if C is to be determined.

The electric field produced by the potential Φ_1 (r, z) and the components of the first-order current can be found by means of (6) and (4).

We consider the case when the radius of the plasma stream is small compared to the characteristic length L^* ($b \ll 1$, thin stream). We expand the modified Bessel functions I_0(r) and I_1(r) into a power series in the small radius $r < b$ and limit ourselves to the first two terms; we thus obtain the following equations for the components of the vector current density:

$$\begin{aligned} j_{1r} &= {}^1/_{16}\, rb^2[1 - (r/b)^2]\,(\sin 2z + B_1 \cos z) \\ j_{1z} &= \left(\frac{b^2}{16} - \frac{2C}{b^2}\right) - \frac{b^2}{8}\left[1 - 2\left(\frac{r}{b}\right)^2\right](\sin z + B_1)\sin z \end{aligned} \tag{14}$$

We many note that only the second of these equations contains the undetermined constant C. In order to see the physical significance of this constant, we introduce the total current I_{1z} along the axis of symmetry:

$$I_{1z} = 2\pi\int_0^b j_{1z}\, r\, dr$$

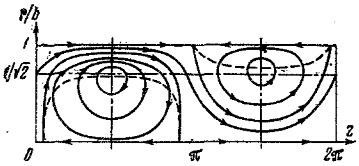

Fig. 2

and C can be expressed linearly in terms of this current as

$$C = -I_{1z}/2\pi + b^4/32 \tag{15}$$

The expressions for the current density components are thus reduced to the form

$$\begin{aligned} j_{1r} &= {}^1/_{16}\, r b^2 \left[1 - (r/b)^2\right] (\sin 2z + B_1 \cos z) \\ j_{1z} &= \frac{I_{1z}}{\pi b^2} - \frac{b^2}{8}\left[1 - 2\left(\frac{r}{b}\right)^2\right] (\sin z + B_1) \sin z \end{aligned} \tag{16}$$

Instead of applying an additional condition on the potential for determining C, we can impose a condition on the current I_{1Z}. We might, for example, assume that there is zero total axial current or $I_{1Z} = 0$. However, since we are not taking into account effects connected with the input and output of the plasma, we can leave open the question of the actual condition imposed.

The differential equation defining the vector lines of current density (16) in the plane $\varphi = \text{const}$ has the following form:

$$\frac{dr}{dz} = \frac{r\left[1 - (r/b)^2\right](2 \sin z + B_1) \cos z}{2\left[M + (2(r/b)^2 - 1)(\sin z + B_1) \sin z\right]} \qquad \left(M = \frac{8 I_{1z}}{\pi b^4}\right) \tag{17}$$

In place of I_{1Z}, we can now talk of possible values for M. It seems best to consider two cases separately: $M = 0$ and $M \neq 0$.

Suppose $M = 0$. Physically this means that there is no total axial current flowing through any arbitrary cross section $z = \text{const}$. The current density lines must therefore close inside the given volume. In fact, a study of the singular points of the vector lines corresponding to (17) shows that in each period the lines form a familiy of closed loops and the number and positions of these depend on the depth of modulation of the magnetic field $h = B_0{}^*/B_1{}^* = 1/B_1$. The schematic form of the current loops in the plane $\varphi = \text{const}$ with $I_{1Z} = 0$ is given in Fig. 1.

It is interesting to follow the change in the structure of the vector field as the modulation depth increases. If h is small ($0 < h < {}^1/_2$), there are two families of closed loops in one period and the centers of these lie on the line $r = b/\sqrt{2}$ at the points $z = \pi/2$ and $z = 3\pi/2$ (Fig. 1a). The directions of the currents are opposite in neighboring families of loops and the boundaries separating the families are represented by the lines $z = 0$, $z = \pi$ and $z = 2\pi$. With further increase in h, the shape in the first half-period $(0, \pi)$ remains the same and all the changes take place in the second half-period $(\pi, 2\pi)$. We may note that for all values of h the centers of the families remain on the line $r = b/\sqrt{2}$. When h passes through the value of ${}^1/_2$, the point $r = b/\sqrt{2}$, $z = 3\pi/2$ becomes a saddle point and the family in the second half-period splits into two parts divided by the line $z = 3\pi/2$. The centers of these families (Fig. 1b) lie at the points

$$z = \pi + \arcsin\ 1/2h, \quad z = 2\pi - \arcsin\ 1/2h.$$

This vector field structure is maintained for ${}^1/_2 < h < 1$. When h passes through the value 1, the point $r = b/\sqrt{2}$, $z = 3\pi/2$ again becomes a center with three families now in the half-period; the third family comes in between the previous two and is separated from them by the lines (Fig. 1c)

$$z = \pi + \arcsin h^{-1}, z = 2\pi - \arcsin h^{-1}.$$

This third family remains as the only one in the half-period $(\pi, 2\pi)$ as $h \to \infty$ and the other two disappear (Fig. 1d). It now becomes meaningless to talk of the modulation of a uniform field because this case corresponds to the value $B_1 = 0$ and there is no uniform field since it changes sign. The spatial period of the current density vector is half that of the magnetic field and is equal to π. This first-order Hall current field has been derived earlier [9] for a plasma with a spatially periodic magnetic field which changes sign.

Suppose $M \neq 0$. If there is a longitudinal current, then the current density vector field is more complicated. For small values of I_{1Z} there are still families of closed loops but the centers are shifted from the positions with $I_{1Z} = 0$ and the loops are deformed. In addition to the loops, there are vector lines which do not close inside a period. At some value of I_{1z}, determined by the modulation depth h, the families of closed loops disappear.

Magnetic fields with small modulation depths (corrugated fields) are of the most interest. We therefore consider in more detail the case where h lies in the interval (0, $\frac{1}{2}$). The singular points of the vector **j** occur where the components j_1r and j_{1z} are both simultaneously equal to zero. For the interval considered, the component j_1r is equal to zero at $z=\pi/2$ and $z=3\pi/2$, and the component j_{1z} is zero on the line whose equation is

$$M + [2 (r / b)^2 - 1] \sin z (\sin z + B_1) = 0 \tag{18}$$

If the value of M admits the existence of a singular point, the nature of this point is the same as that which occurs for $M=0$.

Figure 2 shows schematically the positions of the vector lines for $I_{1z} \neq 0$, $M \neq 0$ and a modulation depth $h < \frac{1}{2}$. The dashed lines depict the lines (18) where $j_{1z}=0$. It can be seen that there are two families of closed loops in a period and that the centers of these are shifted with respect to the case $M=0$ (see Fig. 1a). The shift increases as the value of the longitudinal current I_{1z} goes up. Investigation shows that this closed-loop structure is maintained up to a value $M=(1-h)/h$, where the family in the second half-period disappears. When $M=(1+h)/h$, the family in the first half-period also disappears and all the vector lines remain unclosed.

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